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1994 J. Phys. A: Math. Gen. 27 3139

(http://iopscience.iop.org/0305-4470/27/9/026)

J. Phys. A: Math. Gen. 27 (1994) 3139–3148. Printed in the UK

Reduction of the nonlinear Dirac equation to a nonlinear Schrödinger equation with a correction term

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Received 15 December 1993

Abstract. We examine the low-energy limit of the nonlinear Dirac equation (NLDE) in $1+1$ dimensions with a Lorentz scalar self-interaction. Unlike the nonlinear Schrödinger equation (NLSE), which is integrable, the NLDE is known to exhibit rich dynamics of the soliton–soliton collision when the relative speed of the solitons is small. The NLDE is intrinsically different from the NLSE even when the energy involved is small. When it is modified by adding a specific correction term, however, the NLSE well reproduces the complex features of the soliton–soliton collision described by the NLDE.

1. Introduction

In quantum mechanics the Dirac equation is reduced to the Schrödinger equation in the non-relativistic limit. Is there a similar relationship between the nonlinear Dirac equation (NLDE) and the nonlinear Schrödinger equation (NLSE)? This is the question that motivated this paper. Here it is understood that the mass that appears in these equations is finite. We confine ourselves to $1+1$ dimensions throughout. There are two types of the NLDE, depending on the self-interaction which can be a Lorentz scalar or a vector. We focus on the NLDE with the Lorentz scalar self-interaction, which is more complex and interesting. By the NLDE we mean this equation in the following unless otherwise we indicate.

Long ago Alvarez and Carreras carried out numerical experiments for the NLDE and found solutions which exhibit rich dynamics of soliton–soliton collision [1]. In particular, two solitons can collide to form a quasi-bound state which slowly decays. Interestingly, such complexities occur only when the relative speed of the incident solitons is below a certain critical value. As Alvarez and Carreras pointed out, the intricacy of their results indicates that the NLDE is a non-integrable equation. Let us add that the NLDE is integrable if the mass is zero [2]. On the other hand, the NLSE is one of the well known integrable nonlinear equations (e.g. see [3]). If the NLDE is non-integrable, the NLDE and NLSE are intrinsically different even in the low-energy limit. Then the answer to the question raised in the preceding paragraph is negative.

The soliton–soliton collision that the NLSE describes is simple [3]. The solitons come out of collision with exactly the same shapes and the same speeds with which they entered. This is so for any relative speed of the solitons. This salient feature characterizes the soliton in its strict definition. The NLSE has another interesting type of solution which describes a situation in which two solitons are bound. The two solitons keep oscillating against each other with constant period and amplitude. This bound state is stable [4]. As was found by Alvarez and Carreras, the soliton–soliton collision of the NLDE for small speeds is very different from that of the NLSE. We use the word 'solitons' in the NLDE case even though complex inelastic processes can arise. The NLDE also has solutions which represent soliton–soliton bound states, but these states are unstable and decay [1].

The next question that naturally arises is: what is the non-relativistic limit of the NLDE, if it is not the NLSE? The purpose of this paper is to find an answer to this question. We show that the NLSE with a specific correction term, which we call the modified NLSE in the following, does reproduce the complex features of the NLDE. The correction term is an internal perturbation rather than an external one. Apart from the interest in the NLDE itself, which can be thought of as a model of hadrons, the study of various perturbations to the NLSE may be relevant in applications of the NLSE in diverse fields of physics [5].

In section 2 we present a heuristic method of reducing the NLDE to the modified NLSE. In section 3 we discuss some general features of the modified NLSE. In section 4 we solve the modified NLSE numerically and examine the soliton–soliton collision. The results are discussed in section 5. There are three appendices. In appendix 1 we briefly discuss the NLDE with the self-interaction of the vector type, and in appendices 2 and 3 we give details of some of the formulae of the main text.

2. Non-relativistic reduction of the nonlinear Dirac equation

The NLDE that we examine is

$$\mathrm{i}\psi_t = -\mathrm{i}\alpha\psi_x + \beta m\psi - g(\psi^\dagger\beta\psi)\psi \tag{2.1}$$

where ψ is a two component spinor, $\psi_t = \partial\psi/\partial t$, $\psi_x = \partial\psi/\partial x$, m (>0) is the 'mass' and g is a dimensionless coupling constant which we assume to be positive. Throughout this paper we use natural units such that $c = \hbar = 1$. For the 2×2 Dirac matrices α and β, we take $\alpha = \sigma_y$ and $\beta = \sigma_z$ where the σs are the usual Pauli matrices. Let ψ be

$$\psi(x, t) = \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} \mathrm{e}^{\mathrm{i}mt}. \tag{2.2}$$

Then (2.1) becomes

$$\mathrm{i}u_t = -v_x - g(|u|^2 - |v|^2)u \tag{2.3}$$

$$\mathrm{i}v_t = u_x - [2m - g(|u|^2 - |v|^2)]v. \tag{2.4}$$

We assume that m is much larger than the interaction term involved and rewrite (2.4) as

$$\begin{aligned} v &= \frac{u_x - \mathrm{i}v_t}{2m - g(|u|^2 - |v|^2)} \\ &\simeq \frac{1}{2m}\left(1 + \frac{g}{2m}(|u|^2 - |v|^2)\right)(u_x - \mathrm{i}v_t). \end{aligned} \tag{2.5}$$

We furthermore assume that $|v| \ll |u|$ and $|v_t| \ll |v_x|$. Then (2.5) can be reduced to

$$v \simeq \frac{1}{2m}\left(1+\frac{g}{2m}|u|^2\right)u_x \tag{2.6}$$

which we substitute into (2.3) to find

$$\mathrm{i}u_t \simeq -\frac{1}{2m}u_{xx} - g|u|^2 u - \frac{g}{4m^2}(|u|^2 u_{xx} + u^* u_x^2). \tag{2.7}$$

We admit that the non-relativistic reduction given above is only heuristic because the validity of the assumption of $|v_t| \ll |v_x|$ is not clear. Nevertheless, let us proceed.

3. Modified nonlinear Schrödinger equation

The NLSE that we have mentioned several times already reads as

$$\mathrm{i}\psi_t = -\frac{1}{2m}\psi_{xx} - g|\psi|^2\psi. \tag{3.1}$$

Rewriting u as ψ in (2.7), we now propose to examine the following modified NLSE as the non-relativistic reduction of the NLDE (2.1):

$$\mathrm{i}\psi_t = -\frac{1}{2m}\psi_{xx} - g|\psi|^4\psi - \frac{g}{4m^2}(|\psi|^2\psi_{xx} + \psi^*\psi_x^2). \tag{3.2}$$

The last term is the correction added to the NLSE. It is understood that ψ is normalized as $\int_{-\infty}^{\infty} \mathrm{d}x|\psi(x,t)|^2 = 1$.

Let us discuss some general features of (3.2). The equation can be identified with the Euler–Lagrange equation for the Lagrangian density

$$\mathscr{L} = \mathrm{i}\psi^*\psi_t - \frac{1}{2m}\psi_x^*\psi_x + \frac{g}{2}|\psi|^2 - \frac{g}{4m^2}|\psi\psi_x|^2 \tag{3.3}$$

where the last term is the correction. Suppose one starts with the NLSE (3.1) and one tries to modify it by adding a term which is real and bilinear with respect to ψ and/or ψ^* and also with respect to ψ_x and/or ψ_x^*. There are a few such possibilities, $|\psi\psi_x|^2$, $[(\psi^*\psi_x)^2 + \text{c.c.}]$, $\mathrm{i}[(\psi^*\psi_x)^2 - \text{c.c}]$, etc, where c.c. stands for the complex conjugate of the preceding term. The correction term of (3.3) is the simplest choice.

The correction term of $\mathscr{L}$ is negative, which means that the correction is repulsive. This is consistent with the following aspect regarding the NLSE versus the NLDE. Each of these equations has a single-soliton solution. We summarize the single-soliton solution of the NLSE in the next section. For the NLDE, see [6]. For the same value of g the soliton-binding energy is greater for the NLSE than for the NLDE. This means that the relativistic effect on the binding is repulsive.

The (. . .) of the correction term of (3.2) can be rewritten as

$$|\psi|^2\psi_{xx} + \psi^*(\psi_x)^2 = \psi^*(\psi\psi_x)_x = -(\psi^* p\psi p)\psi \tag{3.4}$$

where $p = -\mathrm{i}\partial/\partial x$. The operator $(\psi^* p\psi p)$ is not Hermitian in the sense that

$$(\psi^* p\psi p)^\dagger = p\psi^* p\psi = (\psi^* p - \mathrm{i}\psi_x^*)(\psi p - \mathrm{i}\psi_x) \neq \psi^* p\psi p. \tag{3.5}$$

It is Hermitian, however, in the sense that

$$\int_{-\infty}^{\infty} \psi^*(\psi^* p \psi p)^\dagger \psi \, \mathrm{d}x = \int_{-\infty}^{\infty} \psi^*(\psi^* p \psi p) \psi \, \mathrm{d}x = 2 \int_{-\infty}^{\infty} \mathrm{d}x |\psi \psi_x|^2. \tag{3.6}$$

Here it is understood that $\psi(x, t) \to 0$ as $x \to \pm\infty$ for any fixed value of t.

The 'charge' defined by

$$Q = \int_{-\infty}^{\infty} \rho(x, t) \, \mathrm{d}x = \int_{-\infty}^{\infty} |\psi(x, t)|^2 \, \mathrm{d}x \tag{3.7}$$

remains t-independent. This is related to the invariance of $\mathscr{L}$ with respect to the phase transformation $\psi \to \mathrm{e}^{\mathrm{i}\lambda}\psi$. The charge conservation is necessary for having soliton solutions. We define the energy E of the system by

$$E = \int_{-\infty}^{\infty} \mathscr{H} \, \mathrm{d}x \tag{3.8}$$

where the Hamiltonian density $\mathscr{H}$ is given by

$$\mathscr{H} = \frac{1}{2m} |\psi_x|^2 - \frac{g}{2} |\psi|^4 + \frac{g}{4m^2} |\psi \psi_x|^2. \tag{3.9}$$

The energy E is also conserved.

Suppose ψ is in the form of a wave packet and define its centre of mass (or charge) by

$$x_c(t) = \int_{-\infty}^{\infty} \rho(x, t) x \, \mathrm{d}x. \tag{3.10}$$

Then it can be shown that

$$\frac{\mathrm{d}x_c}{\mathrm{d}t} = \frac{1}{m} \int_{-\infty}^{\infty} \psi^* \left(1 + \frac{g}{2m} \rho\right) p \psi \, \mathrm{d}x \tag{3.11}$$

$$\frac{\mathrm{d}^2 x_c}{\mathrm{d}t^2} = \frac{g}{2m^3} \int_{-\infty}^{\infty} \left(1 + \frac{g}{2m} \rho\right) \rho_x |\psi_x|^2 \, \mathrm{d}x. \tag{3.12}$$

Some details of the derivation of these equations are given in appendix 2.

Equation (3.12) is similar to but different from the formula known by the name of Ehrenfest's theorem in quantum mechanics [7]. This difference did not surprise us because we were expecting some departure from the non-relativistic relation between velocity and momentum. Because there is no external force acting on the system we expect that $\mathrm{d}^2 x_c/\mathrm{d}t^2 = 0$ holds, which means that the wave packet moves with a constant speed. This is not obvious from (3.12). However, if ρ and $|\psi_x|^2$ are both even functions with respect to the centre of mass of the wave packet, the integral of (3.12) vanishes. This is indeed the case for the one-soliton solution that we found numerically.

4. Soliton–soliton collision

Unlike for (3.1) we have not been able to solve (3.2) analytically. Therefore we have solved (3.2) numerically by means of the standard explicit finite difference method.

Before examining the solutions of (3.2), however, it is convenient to summarize the single-soliton solution of the NLSE (3.1).

Among the many interesting solutions of the NLSE (3.1), the simplest one is the single-soliton solution given by

$$\psi(x,t)=f(x,t;v)$$
$$\equiv\left(\frac{\kappa}{2}\right)^{1/2}\operatorname{sech}[\kappa(x-vt)]\exp\{i[mvx-(\varepsilon_0+\tfrac{1}{2}mv^2)t]\}. \tag{4.1}$$

This is normalized as $\int_{-\infty}^{\infty}|f(x,t;v)|^2\,dx=1$. When $v=0$, $f(x,t;0)$ is the bound state solution of the t-independent equation

$$\varepsilon_0 f=-\frac{1}{2m}f_{xx}-g|f|^2f. \tag{4.2}$$

The constants in (4.1) and (4.2) are related by

$$\kappa=\tfrac{1}{2}mg \qquad \varepsilon_0=-\frac{\kappa^2}{2m}. \tag{4.3}$$

In solving (3.2) we set up the initial condition in terms of $\psi(x,t)$ of (4.1). In all the numerical illustrations in this paper we take $m=1$, $g=1$ and $\kappa=\frac{1}{2}$.

Now let us turn to the modified NLSE (3.2). Before discussing the soliton–soliton collision, we have to make sure that the equation allows single-soliton solutions. Figure 1 shows the charge density $\rho(x,t)=|\psi(x,t)|^2$, which we obtained by starting with

$$\psi(x,t=0)=f(x,0;v) \tag{4.4}$$

where $f(x,t;v)$ is the single-soliton solution (at $t=0$) of (4.1). For v, we took $v=0.1$. If there is no correction term, the $\psi(x,t)$ that starts with $f(x,0;v)$ at $t=0$ is exactly given by $f(x,t;v)$. In the presence of the correction term, the $\psi(x,t)$ for $t>0$ will be different from $f(x,t;v)$. We found, however, that $\psi(x,t)$ quickly settles down to a stationary form which is close to the unperturbed wavefunction $f(x,t;v)$. The solution moves with a constant speed, which is consistent with $d^2x_c/dt^2=0$. The speed of the soliton shown in figure 1 is about 0.5% larger than the input value of v. This is because

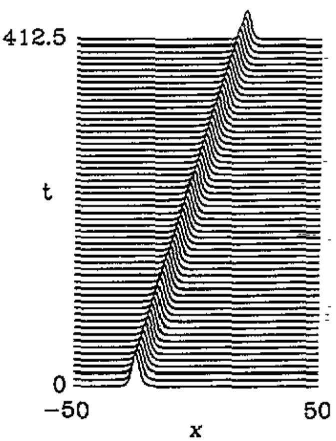

Figure 1. The density $\rho(x,t)=|\psi(x,t)|^2$ of the single-soliton solution of the modified NLSE (3.2); $m=1$, $g=1$, $\kappa=\frac{1}{2}$ and $v=0.1$.

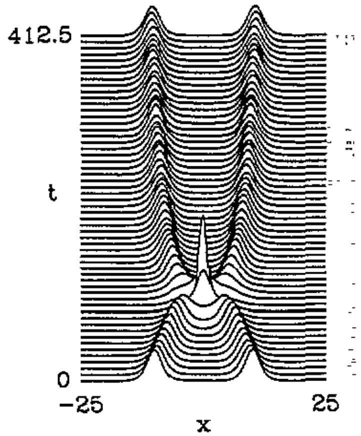

Figure 2. Soliton–soliton collision of the modified NLSE (3.2); $m=1$, $g=1$, $\kappa=\frac{1}{2}$ and $v=0.06$.

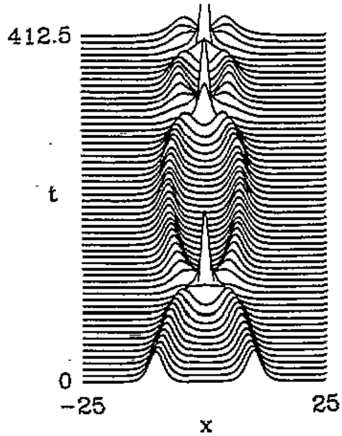

Figure 3. Soliton–soliton collision of the modified NLSE (3.2); $m=1$, $g=1$ and $v=0.05$.

the initial wavefunction $f(x, t; v)$ we used is not an exact solution of (3.2). We confirmed such single-soliton solutions for various input values of v. In appendix 3 we discuss the solution of the stationary state.

Let us now examine the soliton–soliton collision. For the initial condition we assume

$$\psi(x, t=0)=f(x-x_0, 0; -v)+f(x+x_0, 0; v) \tag{4.5}$$

which is a superposition of two (unperturbed) wave packets placed at $x=\pm x_0$. It is understood that $\kappa x_0 \gg 1$ so that the overlap of the two wave packets is negligible. Then $\psi(x, 0)$ is normalized as $\int_{-\infty}^{\infty} |\psi(x, 0)|^2 \, dx=2$, and so is $\psi(x, t)$. In the absence of the correction term the ψ started with (4.5) develops into an analytically known two-soliton solution of the NLSE (3.1). That solution describes the elastic collision process for two solitons with initial velocities $\pm v$. In contrast to that, the correction term that we introduced causes considerable complexity in the collision process.

Figures 2–4 show the charge density $\rho(x, t)=|\psi(x, t)|^2$ for the three cases with $v=$ 0.06, 0.05 and 0.025. There seems to be a critical value v_c, between 0.05 and 0.06, for the initial soliton speed v. For $v>v_c$ (figure 2) the collision is elastic, whereas it becomes

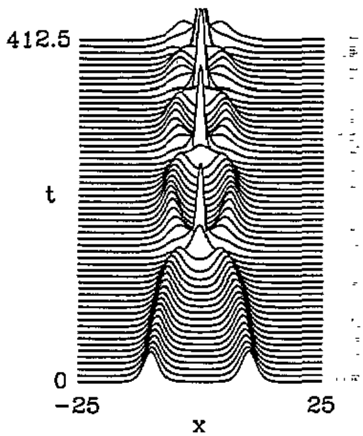

Figure 4. Soliton–soliton collision of the modified NLSE (3.2); $m=1$, $g=1$ and $v=0.025$.

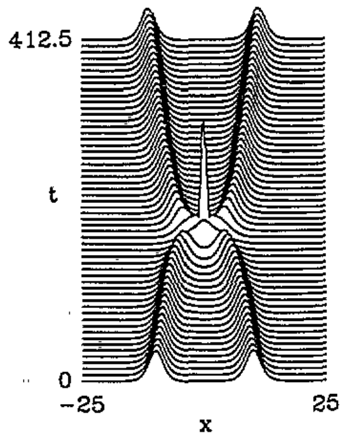

Figure 5. Soliton–soliton collision of the NLSE (3.1); $m=1$, $g=0$ and $v=0.025$.

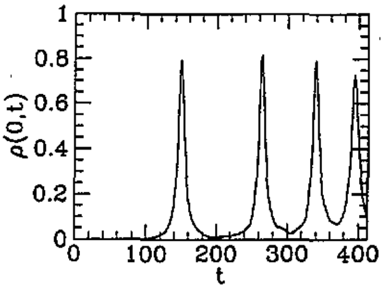

Figure 6. The charge density $\rho(0, t)$ at the centre of the bound state of the modified NLSE (3.2). The parameters are the same as for figure 4.

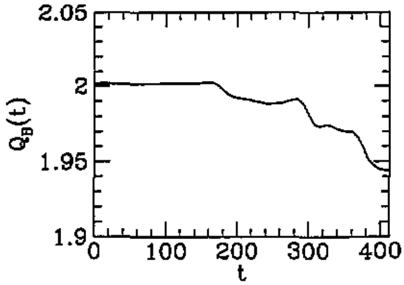

Figure 7. The soliton charge $Q_B(t)$ of (4.6) of the modified NLSE (3.2). The parameters are the same as for figure 4.

inelastic for $v<v_c$ (figures 3 and 4). The elastic collision is very similar to that of the NLSE (3.1) without the correction. The two solitons emerge from collision with the same shapes and speeds with which they entered. For $v<v_c$ the two solitons get bound, and oscillate against each other. However, the period of oscillations gradually decreases in the course of time. Figure 5 is to be compared with figure 4. In these two figures the ψs are subject to exactly the same initial condition but figure 5 shows the result for the NLSE (3.1). The difference between figures 4 and 5 is solely due to the correction term. For the accuracy of our calculations we have checked the conservation of charge Q and energy E, both of which are supposed to be t-independent. For the time intervals shown in all the figures, the error for Q is less than 0.3% while the error for E less than 0.6%.

Let us examine the case of $v=0.025$ in more detail. Figure 6 shows the charge density at $x=0$, $\rho(0, t)$, which oscillates as a function of t. Let us define the charge carried (or contained) by the bound state by

$$Q_B(t)=\int_{-r}^{+r}\rho(x, t)\,dx. \tag{4.6}$$

This quantity depends on the range parameter r. Let us choose somewhat arbitrarily $r=25$. Figure 7 shows that $Q_B(t)$ gradually depreciates. This means that charge (and also energy) is dissipated. In the absence of the correction term such dissipation of course does not occur. In the preceding paragraph we said that inelastic collision occurs for the initial soliton speed $v<v_c$. By 'inelastic' we do not mean that the total energy E decreases, rather we mean that the energy contained in the solitons or their bound state decreases. In figure 7, $Q_B(0)$ is slightly larger than 2. This is because the two terms on the right-hand side of (4.5), which we assume for the initial condition, have a slight overlap.

Finally, figure 8 shows $\rho(x, t)$ for the following situation. Imagine that the correction term is somehow turned off soon after the two solitons have merged and formed a bound state. To be more explicit, let us make the coupling constant g for the correction term t-dependent by the substitution

$$g\rightarrow gf(t)\equiv g\frac{1+\exp(-t_c/t_s)}{1+\exp[(t-t_c)/t_s]} \tag{4.7}$$

where $t_c=165$ and $t_s=5$. We keep the unperturbed part unchanged. The $f(t)$ of (4.7) is turned off at $t\simeq t_c=156$, not abruptly but over a time interval of order t_s. In figure 8 the two solitons meet at $t\simeq150$. Up to $t\simeq165$, figure 8 is practically identical with

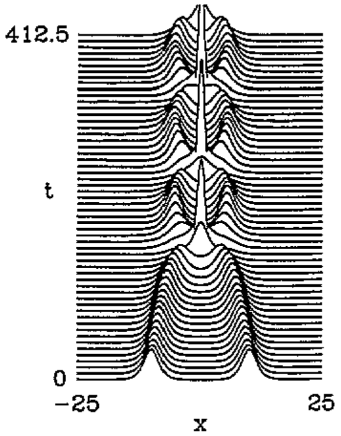

Figure 8. Stabilization of the bound state; $m=1$, $g=1$ and $v=0.25$. The perturbation term has been modified as in (4.7).

figure 4. Note that the oscillating bound state formed by the solitons becomes stable after the correction term is turned off. The period of oscillations becomes constant.

5. Discussion

We have proposed a modified NLSE (3.2) as the non-relativistic reduction of the NLDE (2.1). The soliton–soliton collision described by the modified NLSE exhibits rich dynamics. In particular, when the relative speed of the incident solitons is below a critical value v_c, the merger of the two solitons results in a quasi-bound state which decays slowly. This formation of the quasi-bound state is strikingly similar to what Alvarez and Carreras found for the NLDE [1]. This justifies the non-relativistic reduction of the NLDE which we presented in section 2 in a heuristic manner. Alvarez and Carreras adopted interactions stronger than we assumed. In terms of our notation, their κ is $0.6\sim0.8$ as compared with our $\kappa=0.5$. They found the quasi-bound state of solitons for $v\lesssim 0.06$. This is consistent with our estimate that $0.05\lesssim v_c \lesssim 0.06$.

We noted in section 3 that the correction term of the modified NLSE is repulsive. It is interesting that this repulsive correction pulls two solitons into a bound state. This means that the net effect of the repulsion is somehow reduced when the two solitons are brought close to each other. We have also tried the following. Assume artificially that the g of $-g|\psi|^2\psi$ of the unperturbed part and the g of the correction term are different. Denote these two gs by g_0 and g_1, respectively. Vary the value of g_1 of the correction term but keeping the $g_0=1$ fixed. In this way we found that the complex soliton–soliton interaction is not peculiar to the special choice of $g_1=g_0$. As g_1 is reduced, the effects of the correction term become weaker and the critical velocity v_c becomes smaller. It seems, however, that v_c remains finite as long as $g_1>0$. If $v_c>0$, the modified NLSE (3.2) is qualitatively different from the unperturbed NLSE (3.1) as far as the soliton–soliton interaction is concerned. In this sense, the effects of the correction term of the modified NLSE are not simply perturbative.

Acknowledgments

We thank Professor T Tsuboi for helpful discussions. This work was supported by the Ministry of Education of Japan and the Natural Sciences and Engineering Research Council of Canada.

Appendix 1

Let us briefly discuss the NLDE with the self-interaction of the Lorentz vector type. This equation appears in the Thirring model of quantum field theory, and is known to be integrable [8]. Let us call this equation the Thirring equation. Below (3.2) we discussed other possible forms of the perturbation term in the Lagrangian density. Instead of $(g/4m^2)|\psi^*\psi_x|^2$, suppose we choose

$$\mathscr{L}'=\frac{g}{8m^2}[(\psi^*\psi_x)^2+\text{c.c.}]. \tag{A1.1}$$

Then the NLSE is modified to

$$\mathrm{i}\psi_t=-\frac{1}{2m}\psi_{xx}-g|\psi|^2\psi+\frac{g}{4m^2}[\psi^*_{xx}\psi^2+2|\psi_x|^2\psi-\psi^*(\psi_x)^2]. \tag{A1.2}$$

The above equation can be related to the Thirring equation

$$\mathrm{i}\psi_t=-\mathrm{i}\alpha\psi_x+\beta m\psi-g[\psi^\dagger\psi-(\psi^\dagger\alpha\psi)\alpha]\psi \tag{A1.3}$$

in a manner similar to the manipulation done in section 2.

We have examined soliton–soliton collision described by (A1.2) with the same values of the parameters and the same initial conditions as those for (3.2). We found no quasi-bound state of two solitons in this case. As far as we saw the soliton–soliton collision is always elastic irrespective of the incident relative speed of the solitons. This is consistent with the interpretation that (A1.2) is an approximation to the Thirring equation (A1.3) which is integrable [7].

Appendix 2

Let us give some details of the derivation of (3.11) and (3.12). We start with

$$\mathrm{i}\langle\psi^*A\psi\rangle_t=\langle\psi^*Ah\psi\rangle-\langle(h\psi)^*A\psi\rangle \tag{A2.1}$$

where $\langle\ldots\rangle=\int_{-\infty}^{\infty}(\ldots)\,\mathrm{d}x$, $A=A(x,p)$ is an operator, and

$$h=h_0+h_1 \tag{A2.2}$$

$$h_0=\frac{p^2}{2m}-g|\psi|^2 \qquad h_1=\frac{g}{4m^2}\psi^*p\psi p. \tag{A2.3}$$

The h_0 part of (A2.1) can be handled in the same way as in the usual calculations in quantum mechanics. The h_1 part requires special care because, as we pointed out below (3.4), h_1 is not Hermitian in the usual sense; $\langle(h_1\psi)^*A\psi\rangle\neq\langle\psi^*h_1^\dagger A\psi\rangle$. We found the following formula useful:

$$\langle\psi^*Ah_1\psi\rangle-\langle(h_1\psi)^*A\psi\rangle=\frac{g}{4m^2}[\langle\psi^*A\psi^*(p\psi)^2\rangle-\langle(\psi^*p)^2\psi A\psi\rangle]. \tag{A2.4}$$

If $A=x$,

$$\langle\psi^*x\psi^*(p\psi)^2\rangle=-\langle\psi^{*2}x(\psi\psi_x)_x\rangle=\langle\rho(2x|\psi_x|^2+\psi^*\psi_x)\rangle \tag{A2.5}$$

$$\begin{aligned}\langle(\psi^*p)^2\psi x\psi\rangle&=-\langle\psi^*(\psi^*(x\psi^2)_x)_x\rangle=\langle\psi^*_x\psi^*(x\psi^2)_x\rangle\\&=\langle\rho(2x|\psi_x|^2+\psi^*_x\psi)\rangle.\end{aligned} \tag{A2.6}$$

Thus we obtain

$$\langle\psi^* x\psi^* p\psi p\psi\rangle-\langle(\psi^* p)^2\psi x\psi\rangle=\langle\rho(\psi^*\psi_x-\psi_x^*\psi)\rangle=2\langle\rho\psi^*\psi_x\rangle \tag{A2.7}$$

which yields the term proportional to g of (3.11). We have done integration by parts liberally.

If $A=p$, (A2.4) immediately shows that the part of $\langle\psi^* p\psi\rangle_t$ due to h_1 vanishes. It is also easy to see that the h_0 part of $\langle\psi^* p\psi\rangle_t$ is zero. The right-hand side of (3.12) is due to the time derivative of the g term of (3.11).

Appendix 3

Let us examine the stationary solution of (3.2). Assume that

$$\psi(x,t)=\phi(x)\,\mathrm{e}^{-\mathrm{i}\varepsilon t}. \tag{A3.1}$$

Then (3.2) becomes

$$\varepsilon\phi=-\frac{1}{2m}\phi_{xx}-g\phi^3-\frac{g}{4m^2}[\phi^2\phi_{xx}+\phi(\phi_x)^2] \tag{A3.2}$$

which can be rewritten as

$$\varepsilon\phi=-\frac{1}{4m}\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\left(1+\frac{g}{2m}\phi^2\right)(\phi_x)^2\right]-g\phi^3. \tag{A3.3}$$

This leads to

$$(\phi_x)^2\left(1+\frac{g}{2m}\phi^2\right)+m(2\varepsilon+g\phi^2)\phi^2=0 \tag{A3.4}$$

where we have used the condition that ϕ and ϕ_x both vanish when $x\to\pm\infty$. Integrating (A3.4) we find x as a function of ϕ, but this function is so complicated that we have not been able to turn it around to get ϕ as a function of x.

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